7.2 Notes: Matrix Algebra

Matrices

Definition of Matrix

If m and n are positive integers, then an $m \times n$ (read "m by n") matrix is a rectangular array

$$\begin{array}{c c} & \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \cdots & \text{Column } n \end{array} \\ \begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

in which each **entry** a_{ij} of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

The entry in the *I*th row and *j*th column is denoted by the ________________ ________________ notation a_{ij} . For instance, a_{23} refers to the entry in the second row, third column. A matrix that has only one row is called a _____ ___________, and a matrix that has only one column is called a ________________ ________________. A matrix having *m* rows and *n* columns is said to be _______ ____________ _____ *x* _____. If *m* = *n*, then the matrix is square of order *m* x *m* (or *n* x *n*). For a square matrix, the entries a_{11}, a_{22}, a_{33}, . . . are the main diagonal entries.

Example 1: Determine the order of each matrix.

a. [2]

b. [-1 -3 0 ½]

c. $\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$

d. $\begin{bmatrix} -4 & 1 \\ 5 & 0 \\ -3 & 2 \end{bmatrix}$

Example 2: Solve for $a_{11}, a_{12}, a_{21}, and\ a_{22}$ in the following matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

Definition of matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order *m* x *n*, then their sum is the *m* x *n* matrix given by

A + B = $[a_{ij} + b_{ij}]$.

The sum of two matrices of **different** orders is **undefined.**

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an *m* x *n* matrix and *c* is a scalar, then the **scalar multiple** of A by *c* is the *m* x *n* matrix given by

$$\mathrm{cA} = [ca_{ij}].$$

Example 3: For the following matrices, find (a) A + B, (b) A – B, (c) 3A, and (d) 3A – 2B.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \, and \; B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Example 4: Perform the indicated matrix operations. $\mathbf{3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right)}$

Example 5: Solve for X in the equation 3X + A = B where

$$\boldsymbol{A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \; and \; B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}}$$

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an *m* x *n* matrix and $B = [b_{ij}]$ is an *n* x *p* matrix, then the product AB is an *m* x *p* matrix

$$AB = [c_{ij}]$$

Where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$.

Example 6: Find the product AB using $\boldsymbol{A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \; and \; B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}}$**.**

Example: 7 Find the product AB and BA using $\boldsymbol{A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \; and \; B = \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}}$**.**

Example 8: Find the product of each of the following or state that the product is not defined.

a. $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

c. $\begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}$

Definition of Identity Matrix

The *n* x *n* matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of order *n* x *n*** and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Note that an identity matrix must be *square*. When the order is understood to be *n* x *n,* you can denote I_n simply by I.

For example, $AI_n = A \; and \; I_nA = A.$

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & 3 \end{bmatrix}.$$

The Inverse of a Square Matrix

Definition of the Inverse of a Square Matrix

Let $\boldsymbol{A}$ be an *n* x *n* matrix and let $\boldsymbol{I_n}$ be the *n* x *n* identity matrix. If there exists a matrix $\boldsymbol{A^{-1}}$ such that

$AA^{-1} = I_n = A^{-1}A$

Then **A^{-1}** is called the inverse of **A**. The symbol **A^{-1}** is read "*A inverse*."

Example 9: Show that B is the inverse of A, where

a) $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ and B= $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$ and B= $\begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$

If a matrix A has an inverse, then A is called ________________________ or ___________________; otherwise A is called ___________________. A nonsquare matrix **<u>cannot</u>** have an inverse.

The inverse of a 2 x 2 matrix: If A is a 2 x 2 matrix given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is given by:

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ The denominator $ad - bc$ is called the determinant of the 2 x 2 matrix A.

Example 10: If possible, find the inverse of each matrix

a) $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

b) $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

.

c) $A = \begin{bmatrix} 5 & -1 \\ 3 & 4 \end{bmatrix}$